

Algebraic Rainich theory and antisymmetrisation in higher dimensions

G Bergqvist and A Höglund

Matematiska institutionen, Linköpings universitet, SE-581 83 Linköping, Sweden

E-mail: gober@mai.liu.se, anhog@mai.liu.se

Abstract. The classical Rainich(-Misner-Wheeler) theory gives necessary and sufficient conditions on an energy-momentum tensor T to be that of a Maxwell field (a 2-form) in four dimensions. Via Einstein's equations these conditions can be expressed in terms of the Ricci tensor, thus providing conditions on a spacetime geometry for it to be an Einstein-Maxwell spacetime. One of the conditions is that T^2 is proportional to the metric, and it has previously been shown in arbitrary dimension that any tensor satisfying this condition is a superenergy tensor of a simple p -form. Here we examine algebraic Rainich conditions for general p -forms in higher dimensions and their relations to identities by antisymmetrisation. Using antisymmetrisation techniques we find new identities for superenergy tensors of these general (non-simple) forms, and we also prove in some cases the converse; that the identities are sufficient to determine the form. As an example we obtain the complete generalisation of the classical Rainich theory to five dimensions.

PACS numbers: 02.10.Xm, 02.40.-k, 04.20.Cv, 04.50.+h

1. Introduction

Classical Rainich theory is concerned with studying conditions under which a given energy-momentum tensor in four dimensions can be constructed from a Maxwell field. The algebraic part, which here will be called the algebraic Rainich conditions, gives necessary and sufficient constraints on when a given energy-momentum tensor T_{ab} can be constructed from a 2-form, i.e., when there exists a 2-form F_{ab} such that $T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}F_{cd}F^{cd}g_{ab}$. The conditions, first found by Rainich [7], are

$$T_a{}^a = 0 \quad (1)$$

$$T_{ac}T_b{}^c \propto g_{ab} \quad (2)$$

$$T_{ab}u^a v^b \geq 0 \text{ for every pair of future-causal vectors } u^a \text{ and } v^a. \quad (3)$$

The last condition is usually called the dominant energy condition, and it can be replaced by some other energy condition. Using Einstein's equations and the fact that T_{ab} is trace-free, one may replace T_{ab} by the Ricci curvature tensor R_{ab} in these equations. Then one obtains necessary and sufficient conditions on a four dimensional Lorentzian geometry to correspond (algebraically) to an Einstein-Maxwell spacetime. For F_{ab} to satisfy Maxwell's equations, one also adds a differential condition on T_{ab} or R_{ab} . This view of geometrising physical situations, without going to higher dimension than four, was especially developed by Misner and Wheeler [6]. The theory is well developed for four dimensions and 2-forms but conditions have also been developed for massless scalar fields and some other physical situations (see [2] for many references). It is of interest to generalise the results to other cases and higher dimensions for several reasons. Some gravitation theories live in higher dimensions and can therefore not make use of the classical algebraic Rainich conditions. It can also shed some light on the four dimensional case by considering what happens when the number of dimensions is different than four, thereby showing what is special to four dimensions. Another reason is that there is some nice mathematics to be discovered.

In Bergqvist and Senovilla [2] superenergy tensors of simple forms in arbitrary dimension were examined. One of the results found is that the square of the superenergy tensor of any simple form is proportional to the metric. Conversely, it was also shown that if the square of a symmetric tensor is proportional to the metric, then the tensor is, up to sign, the superenergy tensor of a simple p -form. Furthermore, the trace of the tensor determines the value of p , i.e., what kind of form that can be used to construct the superenergy tensor. These results therefore give a complete generalisation of the classical algebraic Rainich theory in the case the condition $T_{ac}T_b{}^c \propto g_{ab}$ is preserved. As the energy-momentum tensors of Maxwell fields and massless scalar fields are special cases of superenergy tensors of p -forms, it seems that superenergy tensors are the natural objects to consider when a more general algebraic Rainich theory is sought.

In this paper we will regain these results and extend considerations to non-simple forms. The problem is naturally divided into two parts. The first part is to find a necessary identity, which replaces $T_{ac}T_b{}^c \propto g_{ab}$, for the superenergy tensor. The second part is to show that this identity is the best possible for that case, or even better, show that the condition together with the energy condition is sufficient.

The first part will be done by antisymmetrising over the indices of the forms involved. In that process, and even in later cases, it turns out that the metric structure

is not relevant. Therefore those parts will be presented without the metric structure of the vector space.

The metric structure is however crucial for the converse and we will be mainly interested in real vector spaces with Lorentz signature. In this case we obtain complete results for 2-forms of rank 4 in arbitrary dimension. From this we immediately find the necessary and sufficient conditions for an energy-momentum tensor in five dimensions to originate in a 2-form, which is a complete generalisation of the classical algebraic Rainich theory to five dimensions. We also discuss cases of higher rank of the 2-form and p -forms with $p \geq 3$. However, it could also be interesting to work out the converse for other signatures, e.g., with a positive definite metric. The first step would have to be to find a replacement of the dominant energy condition.

In section 2 we present some notation, basic definitions, and a theorem for Lorentz metrics corresponding to the spectral theorem for positive definite metrics. In section 3 we use antisymmetrisation to study Rainich theory for simple forms. Then 2-forms of rank 4 are considered in section 4, with some of the calculations presented in Appendix A. More general forms are treated in sections 5 and 6, and we finish with a discussion in section 7.

2. Notation and prerequisites

In this paper we will use the abstract index notation. The letter n will always refer to the number of dimensions. With a tensor we will mean a multilinear mapping on an n -dimensional vector space. If so preferred it can be seen as a tensor field taken pointwise on an n -dimensional manifold. When talking about $(1, 1)$ -tensors we will use an index free notation. Thus T will mean $T^a{}_b$ and T^2 will mean $T^a{}_c T^c{}_b$ etc. The trace $T^c{}_c$ will be written as $[T]$. In the context of tensors any scalar term will have an implicit identity mapping δ^a_b on it. Due to preference the objects in the index free notation can instead of tensors be viewed as linear mappings with the product being composition, or as matrices with the product being matrix multiplication.

The metric free parts are very general. The tensors can be taken over any finite dimensional vector space over any field of zero characteristic (a sufficient condition for the results in this paper). Occasionally we need to extend the field in order to solve for eigenvalues. The reader who so prefers, may just think of vector spaces over real numbers and extend that to complex numbers whenever necessary.

Whenever we talk about the algebraic multiplicity of an eigenvalue we refer to its multiplicity as a solution of the characteristic equation, not its multiplicity as a solution of the equation at hand. The geometrical multiplicity refers to the dimension of the eigenspace associated with the eigenvalue.

A p -form A is a tensor with p indices that is totally antisymmetric. The *rank* of a p -form A is the dimension of the subspace that is spanned by

$$A_{a_1 a_2 \dots a_p} u^{a_2} v^{a_3} \dots w^{a_p} \quad (4)$$

when the vectors u^a, \dots, w^a varies. Equivalently, the rank of a p -form A is the lowest number of linearly independent (dual-)vectors that must be used in order to construct the p -form.

The p -form A is *simple* if it is a product of 1-forms, i.e.,

$$A_{a_1 \dots a_p} = u_{[a_1} \dots w_{a_p]} \quad (5)$$

where u_a, \dots, w_a are 1-forms, and where the brackets $[]$ denote the antisymmetric part.

The superenergy tensor T_{ab} of a p -form $A = A_{c_1 \dots c_p}$, $1 \leq p \leq n$, is given by [2, 8]

$$T_{ab} = T_{ab}\{A\} = \frac{1}{(p-1)!} \left(A_a{}^{c_2 \dots c_p} A_{bc_2 \dots c_p} - \frac{1}{2p} A^{c_1 \dots c_p} A_{c_1 \dots c_p} g_{ab} \right) \quad (6)$$

if the metric g_{ab} is assumed to have (Lorentz) signature $(- + \dots +)$. As shown in [8], for $1 \leq p \leq n-1$, this expression is equivalent to the original definition first presented by Senovilla, also in [8], which involves A and its Hodge dual. Note that the p -form A and its dual $(n-p)$ -form have the same superenergy tensors if $1 \leq p \leq n-1$ [8].

We would now like to have a definition that does not include the metric structure. Raising the first index on T makes the metric disappear from the definition.

$$T^a{}_b = \frac{1}{(p-1)!} \left(A^{ac_2 \dots c_p} A_{bc_2 \dots c_p} - \frac{1}{2p} A^{c_1 \dots c_p} A_{c_1 \dots c_p} \delta_b^a \right). \quad (7)$$

Without a metric there is no way to relate $A_{a_1 \dots a_p}$ to $A^{a_1 \dots a_p}$. We use different letters for the two forms to emphasise this. In order to simplify things further we omit the non-zero coefficient since that will not affect the metric free part of this paper and will only rescale eigenvalues for the metric dependent parts. So, in this paper the *superenergy tensor* will be defined as

$$T^a{}_b = T^a{}_b\{A, B\} = A^{ac_2 \dots c_p} B_{bc_2 \dots c_p} - \frac{1}{2p} A^{c_1 \dots c_p} B_{c_1 \dots c_p} \delta_b^a. \quad (8)$$

The product of two p -forms A and B will be used frequently in this paper, so we introduce the notation

$$P^a{}_b = A^{ac_2 \dots c_p} B_{bc_2 \dots c_p}. \quad (9)$$

Thus the superenergy tensor can now be written

$$T = P - \frac{1}{2p}[P] \quad (10)$$

using index-free notation.

If the signature $(+ - \dots -)$ is used then the omitted coefficient has a sign that depends on p and that will of course not just add a rescaling but will also change the sign of the eigenvalues.

Whenever a metric is present we assume that A and B are equal relative to the metric. That implies that the superenergy tensor is symmetric relative to this metric, i.e., $g_{ac} T^c{}_b = g_{bc} T^c{}_a$.

The fact that T is symmetric relative to a given metric says something about the structure of T . Exactly what it says depends on the metric. If the metric is positive definite in a real vector space then the spectral theorem says that there is a basis of orthogonal eigenvectors of T . Here we are mainly concerned with metrics with Lorentz signature $(- + \dots +)$, and then the situation is a little bit more involved. The following theorem covers that case. The formulation is due to Hall et al. [4] and is slightly adopted to this paper and extended to arbitrary dimension. The Segré types are not used elsewhere in this paper and can here be considered as just a convenient classification.

Theorem 1 *Let V be an n -dimensional vector space over the real numbers, $n \geq 3$, with a Lorentz metric g with signature $(- + \dots +)$. Let T_{ab} be a real symmetric second order tensor on V . Then (with respect to g) T must take one of the Segré types $\{1, 1111\dots\}$, $\{2111\dots\}$, $\{311\dots\}$ and $\{z\bar{z}111\dots\}$ or a degeneracy thereof. The corresponding canonical forms in some appropriate basis (either orthonormal t^a , $(x_i)^a$,*

$i = 2 \dots n$, or null l^a , m^a , $(x_i)^a$, $i = 3 \dots n$, in which the only non-vanishing inner products are respectively, $-t_a t^a = (x_i)_a (x_i)^a = 1$ and $l_a m^a = (x_i)_a (x_i)^a = 1$) are:

(i) for the type $\{1, 111 \dots\}$ either of

$$T_{ab} = \alpha_1 t_a t_b + \sum_{i=2}^n \alpha_i (x_i)_a (x_i)_b \quad (11)$$

or, equivalently using l^a and m^a ,

$$T_{ab} = (\alpha_2 - \alpha_1) l_{(a} m_{b)} + \frac{1}{2} (\alpha_1 + \alpha_2) (l_a l_b + m_a m_b) + \sum_{i=3}^n \alpha_i (x_i)_a (x_i)_b \quad (12)$$

(ii) for the type $\{2111 \dots\}$

$$T_{ab} = 2\beta_1 l_{(a} m_{b)} \pm l_a l_b + \sum_{i=3}^n \beta_i (x_i)_a (x_i)_b \quad (13)$$

(iii) for the type $\{311 \dots\}$

$$T_{ab} = 2\gamma_1 l_{(a} m_{b)} + 2l_{(a} (x_3)_{b)} + \sum_{i=3}^n \gamma_i (x_i)_a (x_i)_b \quad (14)$$

(iv) finally for the type $\{z\bar{z}111 \dots\}$

$$T_{ab} = 2\delta_1 l_{(a} m_{b)} + \delta_2 (l_a l_b - m_a m_b) + \sum_{i=3}^n \delta_i (x_i)_a (x_i)_b \quad (15)$$

where $\alpha_1, \alpha_2, \dots, \delta_n \in \mathbb{R}$ and $\delta_2 \neq 0$.

Case (i) means that there is a basis of eigenvectors. In case (iv) there is a basis of complex eigenvectors, which here means that there is an invariant two dimensional subspace and real eigenvectors. In the other two cases there is a basis of eigenvectors and Jordan-strings. A Jordan-string of a linear mapping $T^a{}_b$ is a sequence of vectors $(e_i)^a$, $i = 1 \dots k$ such that $T^a{}_b(e_i)^b = \lambda(e_i)^a + (e_{i-1})^a$ for $i = 2 \dots k$ and $T^a{}_b(e_1)^b = \lambda(e_1)^a$. If k is maximal then k is said to be the length of the Jordan-string. If the length is one then we call the Jordan-string trivial.

The fact that the forms A and B are equal may under some circumstances imply further conditions. In the case of Lorentz metric in a real vector space the following general result holds for any T defined by (6)

$$T_{ab} u^a v^b \geq 0 \text{ for every pair of future-causal vectors } u^a \text{ and } v^a. \quad (16)$$

As stated above, this is the dominant energy condition (DEC). It is a special case of the so-called dominant property which has been proven for general superenergy tensors with an arbitrary number of indices [1, 8].

It is straightforward to check which cases in Theorem 1 that are consistent with the DEC. In case (i) we shall in what follows only need T on the form (11).

Theorem 2 Assumptions as in Theorem 1. If, in addition, the DEC is satisfied then the following cases are the only possible ones.

(i) for the type $\{1, 111 \dots\}$

$$T_{ab} = \alpha_1 t_a t_b + \sum_{i=2}^n \alpha_i (x_i)_a (x_i)_b \quad (17)$$

(ii) for the type {2111...}

$$T_{ab} = 2\beta_1 l_{(a} m_{b)} + l_a l_b + \sum_{i=3}^n \beta_i (x_i)_a (x_i)_b \quad (18)$$

where $-\alpha_1 \leq \alpha_i \leq \alpha_1$ and $\beta_1 \leq \beta_i \leq -\beta_1$.

3. Simple forms

3.1. Metric free

Lemma 3 Let $P^a{}_b = A^{ac_2\dots c_p} B_{bc_2\dots c_p}$ where A and B are p -forms and A is simple. Then

$$0 = P \left(P - \frac{1}{p} [P] \right). \quad (19)$$

Proof. Using the property that A is simple gives the first equality

$$\begin{aligned} 0 &= (p+1) A^{[c_1\dots c_p} A^{d_2\dots d_p]} B_{c_1\dots c_p} B_{d_2\dots d_p} \\ &= A^{c_1\dots c_p} B_{c_1\dots c_p} A^{ad_2\dots d_p} B_{bd_2\dots d_p} \\ &\quad - p A^{ac_2\dots c_p} B_{c_1c_2\dots c_p} A^{c_1d_2\dots d_p} B_{bd_2\dots d_p}. \end{aligned} \quad (20)$$

Translating this into index free notation gives the lemma. \square

Theorem 4 If A and B are p -forms and A is simple then $T = T^a{}_b \{A, B\}$ satisfies

$$T^2 = \frac{1}{(2p)^2} [P]^2 \quad (21)$$

which, if $n \neq 2p$, can be written as

$$T^2 = \frac{1}{(n-2p)^2} [T]^2. \quad (22)$$

This is a more general result than the one presented in [2] as only A is assumed to be simple.

The following Corollary follows by repeatedly multiplying equation (22) by T and taking traces.

Corollary 5 If A and B are p -forms, $n \neq 2p$ and A is simple then $T = T^a{}_b \{A, B\}$ satisfies

$$[T^k] = \begin{cases} \frac{n}{(n-2p)^k} [T]^k & \text{if } k \text{ is even} \\ \frac{1}{(n-2p)^{k-1}} [T]^k & \text{if } k \text{ is odd.} \end{cases} \quad (23)$$

3.2. Lorentz metric

We now assume that T is symmetric, and satisfies $T^2 = f$ and the DEC.

That T satisfies $T^2 = f$ gives that T have the two eigenvalues $\pm\sqrt{f}$. Theorem 2 shows that these must be real, which means that $f \geq 0$.

If $f = 0$ then either we have case (i) in Theorem 2 with all coefficients zero implying $T = 0$, or we have case (ii) with all coefficients zero. The 1-form l_a gives this T . The p -form obtained from l_a and $p - 1$ vectors orthogonal to l_a also gives this T .

If $f \neq 0$ then we have case (i) with $\alpha_1 = +\sqrt{f}$ and $\alpha_i = \pm\sqrt{f}$ for $i = 2 \dots n$. This means that we have two non-degenerate eigenspaces: a space-like eigenspace with eigenvalue $+\sqrt{f}$ and an eigenspace containing a time-like direction with eigenvalue $-\sqrt{f}$. Constructing a form from a basis of any of these two eigenspaces and scaling it appropriately gives this particular T .

If k is the dimension of the eigenspace containing the time-like direction, then $[T] = (n - 2k)\sqrt{f}$; the proportionality constant f and the trace of T give the possible values on p , i.e., $p = k$ or $p = n - k$.

We have thus proved of the following theorem.

Theorem 6 *A real symmetric tensor T of a real vector space with signature $(-+ \dots +)$ is the superenergy tensor of a simple form if and only if*

- a) T satisfies DEC
- b) $T^2 = f$

Further, if $f = 0$ then the form is a null-form and if $f \neq 0$ then the form can be chosen as a space-like $\frac{1}{2}(n - [T]/\sqrt{f})$ -form or as a $\frac{1}{2}(n + [T]/\sqrt{f})$ -form containing a time-like direction.

This theorem was first proved in [2], but the proof presented here is both simpler and shorter. In the next section we will use our new methods to generalise the theorem to cases with non-simple forms.

4. 2-forms of rank four

4.1. Metric free

Theorem 7 *Let A and B be 2-forms, $\text{rank } A \leq 4$, $n \neq 4$ and $T = T^a{}_b\{A, B\}$. Then*

$$\left(T^2 - \frac{1}{4}[T]^2 + \frac{1}{4(n-4)}[T]^2\right)\left(T - \frac{1}{n-4}[T]\right) = 0. \quad (24)$$

Proof. Since A has rank at most four the following holds,

$$0 = A^{a[c} A^{de} A^{fg]} B_{bc} B_{de} B_{fg}. \quad (25)$$

Expanding the antisymmetry and collecting the forms together appropriately gives

$$0 = P^3 - \frac{1}{2}[P]P^2 - \frac{1}{4}[P^2]P + \frac{1}{8}[P]^2P. \quad (26)$$

It is possible to rewrite this with T instead of P provided that the number of dimensions is not four,

$$\begin{aligned} 0 = & T^3 - \frac{1}{n-4}[T]T^2 - \frac{1}{4}[T^2]T + \frac{1}{4(n-4)}[T]^2T \\ & + \frac{1}{4(n-4)}[T^2][T] - \frac{1}{4(n-4)^2}[T]^3. \end{aligned} \quad (27)$$

Factoring this gives the theorem. \square

Solving equation (24) is a little bit more involved than the previous case; see Appendix A for details. Table 1 summarises the solutions in terms of eigenvalues and Jordan strings for the product P and the superenergy tensor T (which is just a shift in the eigenvalues but otherwise the same structure). While solving the equation it might be necessary to extend the field of numbers in order for the square roots to exist.

Table 1. Solutions of equation (24)

Eigenvalues of T	Eigenvalues of P	Alg. mult.	Max. length Jordan-str.
Case a: $\frac{1}{n-4}[T]^2 \neq [T^2] \neq \frac{n}{(n-4)^2}[T]^2$			
$+\frac{1}{2}\sqrt{[T^2] - \frac{1}{n-4}[T]^2}$	$-\frac{1}{n-4}[T] + \frac{1}{2}\sqrt{[T^2] - \frac{1}{n-4}[T]^2}$	2	1
$-\frac{1}{2}\sqrt{[T^2] - \frac{1}{n-4}[T]^2}$	$-\frac{1}{n-4}[T] - \frac{1}{2}\sqrt{[T^2] - \frac{1}{n-4}[T]^2}$	2	1
$\frac{1}{n-4}[T]$	0	$n-4$	1
Case b: $\frac{1}{n-4}[T]^2 = [T^2] \neq \frac{n}{(n-4)^2}[T]^2$			
0	$-\frac{1}{n-4}[T]$	4	2
$\frac{1}{n-4}[T]$	0	$n-4$	1
Case c: $\frac{n}{(n-4)^2}[T]^2 = [T^2] \neq 0$			
$-\frac{1}{n-4}[T]$	$-\frac{2}{n-4}[T]$	2	1
$\frac{1}{n-4}[T]$	0	$n-2$	2
Case d: $[T] = 0 = [T^2]$			
0	0	n	3

4.2. Lorentz metric

Assume now that we have a vector space over the real numbers with a metric with Lorentz signature, and that T is symmetric with respect to this metric and satisfies the DEC and equation (24). Then the assumptions in Theorem 2 are satisfied and we can compare the solutions given above with the two cases in the theorem. We want to show that if these conditions are satisfied then there are forms that gives this particular superenergy tensor.

Solution (a) corresponds to case (i) in the theorem. All eigenvalues are real so $[T^2] - [T]^2/(n-4) > 0$. Then there are two cases: $[T^2] > n[T]^2/(n-4)^2$ and $[T^2] < n[T]^2/(n-4)^2$. In the first case $\frac{1}{2}\sqrt{[T^2] - [T]^2/(n-4)} > |[T]/(n-4)|$ which

means that the timelike eigenvalue is $-\frac{1}{2}\sqrt{[T^2] - [T]^2/(n-4)}$. Numbering the other vectors in the theorem such that $(x_2)^a$ has the same eigenvalue as the timelike vector and $(x_3)^a$ and $(x_4)^a$ have the other eigenvalue with multiplicity two, then we can find that one form that gives this superenergy tensor is

$$\begin{aligned} & 2\sqrt{\frac{1}{n-4}[T] + \frac{1}{2}\sqrt{[T^2] - \frac{1}{n-4}[T]^2}} t_{[a}(x_2)_{b]} \\ & + 2\sqrt{-\frac{1}{n-4}[T] + \frac{1}{2}\sqrt{[T^2] - \frac{1}{n-4}[T]^2}} (x_3)_{[a}(x_4)_{b]}. \end{aligned} \quad (28)$$

In the other case, $[T^2] < n[T]^2/(n-4)^2$, the theorem gives $\frac{1}{2}\sqrt{[T^2] - [T]^2/(n-4)} < |[T]/(n-4)|$ which means that the timelike eigenvalue must be $[T]/(n-4)$ and that that $[T]$ must be negative. Numbering the vectors in the theorem appropriately shows that T is obtained from

$$\begin{aligned} & 2\sqrt{-\frac{1}{n-4}[T] + \frac{1}{2}\sqrt{[T^2] - \frac{1}{n-4}[T]^2}} (x_2)_{[a}(x_3)_{b]} \\ & + 2\sqrt{-\frac{1}{n-4}[T] - \frac{1}{2}\sqrt{[T^2] - \frac{1}{n-4}[T]^2}} (x_4)_{[a}(x_5)_{b]}. \end{aligned} \quad (29)$$

Solution (b) in the case where there are only trivial Jordan-strings gives case (i) in the theorem. The timelike eigenvalue must be $[T]/(n-4)$ and $[T]$ must be negative. One form that gives this superenergy tensor is

$$2\sqrt{-\frac{1}{n-4}[T]} \left((x_2)_{[a}(x_3)_{b]} + (x_4)_{[a}(x_5)_{b]} \right). \quad (30)$$

Solution (b) in the case where there are non-trivial Jordan-strings corresponds to case (ii) in the theorem. However, the eigenvalue β_1 that permits Jordan-strings in solution (b) is zero and that is not compatible with case (ii) in the theorem, i.e., the DEC is not satisfied.

Solution (c) in the case where there are only trivial Jordan-strings gives case (i) in the theorem. The timelike eigenvalue is $[T]/(n-4)$ if $[T]$ is negative and $-[T]/(n-4)$ if $[T]$ is positive. In the first case the form

$$2\sqrt{-\frac{2}{n-4}[T]} t_{[a}(x_2)_{b]} \quad (31)$$

gives this superenergy tensor and in the second case

$$2\sqrt{\frac{2}{n-4}[T]} t_{[a}(x_2)_{b]} \quad (32)$$

gives it.

Solution (c) in the case where there is a non-trivial Jordan-string gives case (ii) in the theorem. Then $[T]$ must be negative and $[T]/(n-4)$ is the null eigenvalue. The form

$$2t_{[a}(x_3)_{b]} + 2\sqrt{-\frac{2}{n-4}[T]} (x_4)_{[a}(x_5)_{b]} \quad (33)$$

gives this T .

Solution (d) in the case where there are only trivial Jordan-strings gives that $T = 0$.

Solution (d) in the case where there are non-trivial Jordan-strings gives case (ii) in the theorem (which excludes Jordan-strings of length three) where all β_i are zero. The form

$$2l_{[a}(x_2)_{b]} \quad (34)$$

gives this superenergy tensor.

Thus we have proved the following theorem.

Theorem 8 *A real symmetric tensor $T = T^a_b$ in an n -dimensional real vector space, $n > 4$, with Lorentz signature is the superenergy tensor of a 2-form of rank at most four if and only if*

- a) T satisfies DEC
- b) $\left(T^2 - \frac{1}{4}[T^2] + \frac{1}{4(n-4)}[T]^2\right)\left(T - \frac{1}{n-4}[T]\right) = 0.$

Further, if $(n-4)^2[T^2] < n[T]^2$ then the 2-form is the sum of two simple spacelike 2-forms, and if $(n-4)^2[T^2] > n[T]^2$ then the 2-form is the sum of one simple spacelike 2-form and one simple 2-form containing a timelike direction.

Since the dual of a p -form gives the same superenergy tensor as the p -form itself we have the following corollary.

Corollary 9 *A real symmetric tensor $T = T^a_b$ in an n -dimensional real vector space, $n > 4$, with Lorentz signature is the superenergy tensor of a $(n-2)$ -form which is a sum of at most two simple $(n-2)$ -forms if and only if*

- a) T satisfies DEC
- b) $\left(T^2 - \frac{1}{4}[T^2] + \frac{1}{4(n-4)}[T]^2\right)\left(T - \frac{1}{n-4}[T]\right) = 0.$

Further, if $(n-4)^2[T^2] < n[T]^2$ then the $(n-2)$ -form is the sum of two simple $(n-2)$ -forms containing a timelike direction, and if $(n-4)^2[T^2] > n[T]^2$ then the $(n-2)$ -form is the sum of one simple spacelike $(n-2)$ -form and one simple $(n-2)$ -form containing a timelike direction.

4.3. Five dimensions, Lorentz metric

All cases in five dimensional spaces with a Lorentz metric are now covered; 1-forms and 4-forms are always simple, 2-forms cannot have higher rank than four and 3-forms can be referred back to 2-forms by taking duals of them. Hence, an immediate consequence of Theorem 8 is

Theorem 10 *A real symmetric tensor $T = T^a_b$ in an 5-dimensional real vector space with Lorentz signature is the superenergy tensor of a 2-form if and only if*

- a) T satisfies DEC
- b) $\left(T^2 - \frac{1}{4}[T^2] + \frac{1}{4}[T]^2\right)\left(T - [T]\right) = 0.$

This theorem is the generalisation of the classical algebraic Rainich theory to five dimensions. It gives necessary and sufficient conditions on T to correspond algebraically to the energy-momentum tensor of an electromagnetic field. Note that the difference with four dimensions is the factor $T - [T]$, and that the trace of T , which is non-zero in five dimensions, is built into the equation.

4.4. Four dimensions, metric free

Four dimensions were excluded in Theorem 7. That case was already discussed in detail in Edgar and Höglund [3]. It turns out that the interesting identity is a dimensionally dependent identity. Antisymmetrising over five indices, one more than the number of dimensions, gives the identity

$$0 = \delta_b^{[a} A^{cd} A^{ef]} B_{cd} B_{ef}. \quad (35)$$

Expanding the antisymmetry and substituting for T gives the following theorem.

Theorem 11 *If A and B are 2-forms in four dimensions then $T = T^a{}_b\{A, B\}$ satisfies*

$$T^2 = \frac{1}{4}[T^2]. \quad (36)$$

Combining this with Theorem 4 shows that the square of all superenergy tensors of forms are proportional to the identity mapping in four dimensions.

Note that also Lovelock [5] used antisymmetrising and two different forms A and B to study this case.

5. 2-forms of higher rank

It is always possible to get an identity for 2-forms of at most a given rank k by antisymmetrising over $k+1$ indices in the following way.

$$0 = A^{[c_1} A^{c_2 c_3} \dots A^{c_k c_{k+1}]} B_{bc_1} B_{c_2 c_3} \dots B_{c_k c_{k+1}}. \quad (37)$$

This will give an identity of order $k/2 + 1$ (note that the rank of a 2-form is always even).

In the special case where $n = k$ we can get a dimensionally dependent identity of order $k/2$ by antisymmetrising over $n+1$ indices in the following way.

$$0 = \delta_b^{[a} A^{c_1 c_2} \dots A^{c_{k-1} c_k]} B_{c_1 c_2} \dots B_{c_{k-1} c_k}. \quad (38)$$

It is not possible to get a lower order identity for any of these cases. To see this, take k linearly independent vectors $(v_i)^a$ and k dual vectors $(\omega^i)_a$ such that

$$(\omega^i)_a (v_j)^a = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (39)$$

and define 2-forms A and B by

$$A^{ab} = \alpha_1 (v_1)^{[a} (v_2)^{b]} + \dots + \alpha_{k/2} (v_{k-1})^{[a} (v_k)^{b]} \quad (40)$$

$$B_{ab} = \alpha_1 (\omega^1)_{[a} (\omega^2)_{b]} + \dots + \alpha_{k/2} (\omega^{k-1})_{[a} (\omega^k)_{b]} \quad (41)$$

where all α_j^2 , which will be the eigenvalues of the product P , are chosen distinct. In the first case P will also have the eigenvalue zero. Thus, in the two cases P has $k/2+1$ and $k/2$ different eigenvalues respectively. So will T and therefore cannot satisfy any polynomial equation of lower order.

When a metric is present an orthonormalised basis can of course be used instead of the basis presented above.

It is possible to obtain the identities (37) and (38) recursively. The following theorem tells us how to do that.

Theorem 12 Let

$$\widehat{\text{Id}}_k = \delta_b^{[a} A^{c_1 c_2} \dots A^{c_{k-1} c_k]} B_{c_1 c_2} \dots B_{c_{k-1} c_k} \quad (42)$$

and

$$\text{Id}_k = A^{[c_1} A^{c_2 c_3} \dots A^{c_k c_{k+1}]} B_{bc_1} B_{c_2 c_3} \dots B_{c_k c_{k+1}}. \quad (43)$$

Then the following identities hold.

$$\widehat{\text{Id}}_k = -\frac{k}{k+1} \left(\text{Id}_{k-2} - \frac{1}{k} [\text{Id}_{k-2}] \right) \quad (44)$$

$$\text{Id}_k = -\frac{k}{k+1} P \left(\text{Id}_{k-2} - \frac{1}{k} [\text{Id}_{k-2}] \right) \quad (45)$$

$$\text{Id}_k = -\frac{k}{k+1} \left(T \text{Id}_{k-2} - \frac{1}{n-4} [T] \text{Id}_{k-2} - \frac{1}{k} [\text{Id}_{k-2}] T + \frac{1}{k(n-4)} [T][\text{Id}_{k-2}] \right) \quad (46)$$

Proof. Consider

$$\widehat{\text{Id}}_k = \delta_b^{[a} A^{c_1 c_2} \dots A^{c_{k-1} c_k]} B_{c_1 c_2} \dots B_{c_{k-1} c_k} \quad (47)$$

and split the right hand side into the two cases where in the first case the index a is on the delta and in the second case a is not on the delta.

$$(k+1) \widehat{\text{Id}}_k = \delta_b^a A^{[c_1 c_2} \dots A^{c_{k-1} c_k]} B_{c_1 c_2} \dots B_{c_{k-1} c_k} \\ - k A^{[a c_2} \dots A^{c_{k-1} c_k]} B_{bc_2} \dots B_{c_{k-1} c_k}. \quad (48)$$

By using

$$A^{[c_1 c_2} \dots A^{c_{k-1} c_k]} = A^{c_1 [c_2} \dots A^{c_{k-1} c_k]} \quad (49)$$

we get

$$(k+1) \widehat{\text{Id}}_k = \delta_b^a A^{c_1 [c_2} \dots A^{c_{k-1} c_k]} B_{c_1 c_2} \dots B_{c_{k-1} c_k} \\ - k A^{a [c_2} \dots A^{c_{k-1} c_k]} B_{bc_2} \dots B_{c_{k-1} c_k} \quad (50)$$

which is the same as the identity (44).

The next identity is obtain in a similar way, by starting with

$$\text{Id}_k = A^{[c_1} A^{c_2 c_3} \dots A^{c_k c_{k+1}]} B_{bc_1} B_{c_2 c_3} \dots B_{c_k c_{k+1}} \quad (51)$$

and splitting the right hand side into the two cases where the index c_1 is on the first form and the case where it is not on the first form.

$$(k+1) \text{Id}_k = A^{ac_1} B_{bc_1} A^{[c_2 c_3} \dots A^{c_k c_{k+1}]} B_{c_2 c_3} \dots B_{c_k c_{k+1}} \\ - k A^{ac_3} B_{c_2 c_3} A^{[c_2 c_1} A^{c_4 c_5} \dots A^{c_k c_{k+1}]} B_{bc_1} \dots B_{c_k c_{k+1}}. \quad (52)$$

Using identity (49) gives identity (45).

By substituting $P = T - [T]/(n-2p)$ in identity (45) we obtain identity (46). \square

The coefficient $-\frac{k}{k+1}$ can safely be ignored if the theorem is used to examine the identities $\text{Id}_k = 0$ and $\widehat{\text{Id}}_k = 0$. The expressions Id_k and $\widehat{\text{Id}}_k$ can of course be expressed with P or T as preferred.

The theorem can be used for practical calculations in obtaining identities recursively for 2-forms of higher rank, where the identity for simple forms in Lemma 3 or Theorem 4 can be used as a starting point or even go one step further back to $\text{Id}_0 = P = T - [T]/(n-4)$.

It also helps us to understand the structure of the identities for 2-forms. The interesting case for the identity (44) is when $k = n$. Then it is clear from Theorem 12 that the left hand side of the identity $\widehat{\text{Id}}_n = 0$, given by (44), is trivially trace free, i.e., it is not possible to express $[T^{n/2}]$ in other traces by taking the trace of this identity. The left hand side of this identity will contain a term $[T^{n/2}]/n$ which prevents us from factorising the identity.

The identity (44) also helps us understand why rank four 2-forms did not turn out to be a special case in four dimension. We can see that the recursion only changes the coefficient of the constant term when going to rank n . For four dimensions that did not change the structure of the identity; the factorisation merely relied on the coefficient to be positive (in the real case). In higher dimensions, we will have higher order polynomials and a change in the constant term will have a larger impact on the factorisation.

6. p -forms, $3 \leq p \leq n - 3$

The method of antisymmetrising over the indices of the forms gets more complicated for p -forms when $p \geq 3$. If we antisymmetrise over at least two indices on one form and over at least three forms then it happens that the indices that should contract with the indices of one form are spread out over three or more forms. Then it is not possible to combine the forms in such a way that they can be written in terms of the product P .

Other possibilities fail in the same way, i.e., the forms cannot be combined so that they can be written in terms of the product P . The best way we can do is to antisymmetrise over one index on each form. This can be written in terms of P directly as

$$0 = P^{[a}{}_b P^{c_1}{}_{c_1} \dots P^{c_k]}{}_{c_k} \quad (53)$$

where the form A in the product P is of rank at most k . For the special case when $n = k$ we can get one order lower by

$$0 = \delta_b^{[a} P^{c_1}{}_{c_1} \dots P^{c_k]}{}_{c_k} \quad (54)$$

which just gives us the Cayley-Hamilton equation.

This complication is not just a break down of the method of getting the identities by antisymmetrising but, as the following example shows, a complication in the structure of the product of the forms.

Assume we have a vector space over the real numbers and a metric with Lorentz signature or a positive definite metric. Let $(e_i)^a$, $i = 1 \dots n$ be an orthonormalised basis with $(e_1)^a$ as the timelike base vector in the Lorentz case. Take the vectors

$$\begin{aligned} x^a &= (e_1)^a + (e_2)^a & y^a &= (e_3)^a + 2(e_4)^a & z^a &= (e_5)^a + 3(e_6)^a \\ u^a &= (e_1)^a - (e_2)^a & v^a &= (e_3)^a - 2(e_4)^a & w^a &= (e_5)^a - 3(e_6)^a \end{aligned} \quad (55)$$

and build the form

$$A^{abc} = x^{[a} y^b z^{c]} + u^{[a} v^b w^{c]}. \quad (56)$$

Use the metric to lower the indices on A to get B . The base vectors $(e_i)^a$, $i = 1 \dots 6$, are all eigenvectors of the product $P^a{}_b = A^{acd} B_{bcd}$ with distinct non-zero eigenvalues. If the dimension is more than six then $(e_i)^a$, $i = 7 \dots n$, are eigenvectors with eigenvalue zero. Thus, this P , and therefore the corresponding T , can not satisfy a polynomial identity of lower order than six respectively seven.

This example does of course not rule out the possibility of special cases where it might be possible to obtain something better than the identities (53) and (54).

7. Discussion

The main point in this paper is to highlight the relation between identities from antisymmetrisation and Rainich theory. We have shown how to obtain such identities in several cases and in some cases also shown that they are sharp in the sense that they, together with the DEC, are sufficient conditions for those cases. In some other cases we have argued that there is no better algebraic identity without showing that everything which satisfies the particular identity is also a superenergy tensor of that type.

As we have seen in this paper the problem naturally divides into two parts: (*i*) obtaining identities, and (*ii*) showing that the identity so obtained is the best possible or, even stronger, show that it characterises that class of superenergy tensors.

It is interesting that the first part (if the energy condition is not included) is independent of the metric. The generalisations come “for free” once this fact is respected. Nothing would have been simpler by restricting to real vector spaces with some special metric and assuming that A and B were equal with respect to that metric. Actually, the absence of a metric prevents us from unnecessary raising or lowering indices and thereby helps us to avoid complicating the derivations unnecessarily.

The antisymmetrisations that are the foundations for the identities have different roles in the identities obtained in this paper. In most cases the antisymmetrisation is over more indices than the rank of the tensors involved. On some occasions the number of indices that is antisymmetrised over is larger than the number of dimensions and the identities can thus be viewed as dimensionally dependent identities. When this happens it is not important that the indices antisymmetrised over belong to a particular tensor but can be on any tensor. Normally this is used to replace one pair of forms by a delta and thus reducing the order of the final identity. However, when the rank of the tensors involved is strictly less than the number of dimensions there is nothing to gain from antisymmetrising over more indices just to get a dimensionally dependent identity.

It is surprising that it is only the structure of one of the forms A and B , the simpler one if given a choice, that matters. The forms always have the same structure when constructing superenergy tensors so it is hard to see that this will have any practical impact. However, from a mathematical point of view it is interesting.

A natural question to ask is if it is possible to find forms A and B for any tensor T that satisfies any of the identities obtained in this paper. In other words, does the energy condition together with the symmetry assumption correspond to the requirement that A and B should be equal with respect to a given metric? It is obvious that we get more tensors T when A and B are allowed to be independent. One way to see this is to take any T that satisfies the energy condition and then change sign on one of the forms. The new superenergy tensor so obtained does not satisfy the energy condition except in special cases.

However, there are tensors T which satisfy an identity without being obtainable as a superenergy tensor. One such tensor in five dimensions is the one with a Jordan-string of length three and all eigenvalues zero. The fact that the trace is zero implies that the tensor T and the product of the forms are identical. This tensor satisfies the identity in Theorem 7 but it does not satisfy the identity in Theorem 4 so neither A

nor B can be simple. This means that the forms must be 2-forms or 3-forms. Those two cases are equivalent in five dimensions so we may assume 2-forms, which is also what we would expect since Theorem 7 refers to 2-forms. Since the 2-forms cannot be simple they must both be of rank 4 which means that the dimension of the kernel is one. The kernel of the composition of these two forms, and thus the tensor T , has at most dimension two. This is in contradiction with the assumed T whose kernel has dimension three.

Another question which comes naturally is whether we could have avoided the DEC in Theorem 8 by allowing that either of $\pm T$ is a superenergy tensor as was done in [2]. This is not the case here as is easy to see by considering, e.g., the solution of equation (24) where all eigenvalues are zero and with one Jordan-string of length three. This solution is compatible with a symmetric T according to Theorem 1. However, neither of $\pm T$ is compatible with the DEC according to Theorem 2.

We have seen that the complexity of the algebraic Rainich theory increases with increasing dimension n , and increasing rank of the p -forms. Furthermore, we have not discussed the non-uniqueness problem; how many forms can have the same superenergy tensor, a problem which has been solved in four dimensions (see, e.g., [2]). There are indications that there is a higher degree of uniqueness if $n \neq 2p$ than if $n = 2p$. One may also develop a Rainich theory for superenergy tensors with more indices, such as the Bel and Bel-Robinson tensors [8].

Acknowledgments

G.B. is grateful to the (former) Ministerio de Educación, Cultura y Deporte in Spain for support for a sabbatical year 2001 at the University of the Basque Country in Bilbao (SAB1999-0135).

Appendix A. Solving equation (24)

The equation we want to solve is

$$\left(T^2 - \frac{1}{4}[T^2] + \frac{1}{4(n-4)}[T]^2\right)\left(T - \frac{1}{n-4}[T]\right) = 0. \quad (24)$$

It is clear that we have at most three eigenvalues

$$\pm \frac{1}{2} \sqrt{\left[T^2\right] - \frac{1}{n-4}[T]^2} \quad \text{and} \quad \frac{1}{n-4}[T] \quad (\text{A.1})$$

where it might have been necessary to extend the field of numbers in order for the square root to exist. It remains to find the multiplicities of these and to identify the cases when two or all three of them coincides.

Let the algebraic multiplicities of the eigenvalues be a , b and c . Then we find that

$$[T^2] = \frac{n}{4}\left([T^2] - \frac{1}{n-4}[T]^2\right) - \frac{c}{4}\left([T^2] - \frac{n}{(n-4)^2}[T]^2\right). \quad (\text{A.2})$$

Thus $c = n - 4$ is always a solution to this equation. It is the only solution if the last parenthesis is non-zero. Assume this is the case, i.e., $[T^2] \neq n[T]^2/(n-4)^2$. Note that this happens precisely when the last eigenvalue is distinct from the others. Calculating the trace of T one finds

$$[T] = [T] + \frac{a-b}{2} \sqrt{\left[T^2\right] - \frac{1}{n-4}[T]^2}. \quad (\text{A.3})$$

If the square root is non-zero then $a = b$, and $c = n - 4$ implies that $a = b = 2$. In this case all eigenvalues are distinct. If the square root is zero, i.e., $[T^2] = [T]^2/(n - 4)$, then equation (24) can be written as

$$T^2 \left(T - \frac{1}{n-4} [T] \right) = 0. \quad (\text{A.4})$$

This gives that 0 is an eigenvalue which admits a Jordan-string of length at most two. The fact that $c = n - 4$ gives that the algebraic multiplicity of this eigenvalue must be four.

Consider now the case $[T^2] = n[T]^2/(n - 4)^2$. Then equation (24) can be written as

$$\left(T + \frac{1}{n-4} [T] \right) \left(T - \frac{1}{n-4} [T] \right)^2 = 0. \quad (\text{A.5})$$

If $[T]$ is non-zero then there are two distinct eigenvalues, $\pm [T]/(n - 4)$, where the one with $+$ admits non-trivial Jordan-strings of length at most two. Assume that their algebraic multiplicities are $n - a$ and a . Then

$$[T] = \frac{n - 2a}{n - 4} [T] \quad (\text{A.6})$$

which implies that $a = 2$.

If $[T] = 0$ then equation (24) becomes

$$T^3 = 0 \quad (\text{A.7})$$

which gives that all eigenvalues are zero and that there might be Jordan-strings of length at most three.

References

- [1] Bergqvist G 1999 Positivity of general superenergy tensors *Commun. Math. Phys.* **207** 467–479
- [2] Bergqvist G and Senovilla J M M 2001 Null cone preserving maps, causal tensors and algebraic Rainich theory *Class. Quantum Grav.* **18** 5299–5325 gr-qc/0104090
- [3] Edgar S B and Höglund A 2002 Dimensionally dependent tensor identities by double antisymmetrization *J. Math. Phys.* **43** 659–677 gr-qc/0105066
- [4] Hall G S, Rebouças M J, Santos J and Teixeira A F F 1996 On the algebraic structure of second order symmetric tensors in 5-dimensional space-times *Gen. Rel. Grav.* **28** 1107–1113 gr-qc/9511056
- [5] Lovelock D 1973 The algebraic Rainich conditions *Gen. Rel. Grav.* **2** 149–159
- [6] Misner C W and Wheeler J A 1957 Classical physics as geometry *Ann. Phys., NY* **2** 525–603
- [7] Rainich G Y 1925 Electrodynamics in the general relativity theory *Trans. Am. Math. Soc.* **27** 106–136.
- [8] Senovilla J M M 2000 Super-energy tensors *Class. Quantum Grav.* **17** 2799–2841 gr-qc/9906087